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THEORETICAL STUDY OF LIFT AND MOMENT ON OSCILLATING TRIANGULAR AND RELATED WINGS WITH SUPERSONIC EDGES

A Thesis

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In Partial Fulfillment

of the Requirements for the Degree

Master of Aeronautical Engineering

by

Herbert C. Nelson

March 1951

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CHAPTER I

INTRODUCTION

In high speed aircraft design, a knowledge of the air forces and moments that act on various types of oscillating wing planforms is often required. Such knowledge is useful in considering a number of instability problems, among which are wing flutter and low frequency instability of the aircraft involving control surface deflections. The main source of theoretical information has been the solution of the linearized differential equation for compressible flow.

Current trends in high speed aircraft design are toward the utilization of thin wings. Therefore, the linearized theory can be expected to yield results of practical value (except perhaps in the transonic speed range). For instance, in the supersonic speed range, the linearized theory has been successfully employed to predict the forces on a large variety of non-oscillating thin wings.

The problem of finding the air forces on an oscillating or steady thin wing is usually formulated in terms of the disturbance velocity potential for the wing. The disturbance velocity potential is constructed by superposing solutions to the linearized differential equation corresponding to sources, doublets, and higher order singularities. For example, the action of a wing

in two dimensional supersonic flow can be represented by a surface distribution of sources. The disturbance velocity potential for the wing is then obtained by superposing these singular solutions in such a way as to satisfy the boundary conditions at the surface and at infinity.

I. The Problem Considered

Statement of the problem. The purpose of this investigation is (1) to obtain the velocity potential for an apex forward, thin, flat, oscillating "arrow-head" type wing with supersonic edges moving at a constant supersonic speed; and (2) to obtain from the velocity potential the section forces and moments for the "general" arrow-head wing and the total forces and moments for the particular arrow-head wing usually referred to as the delta wing.

Linearized theory will be used throughout this study.

Importance of study. The force and moment expressions that are obtained in this thesis are useful in investigations of various kinds of instability problems associated with high speed flight; e.g. wing flutter, and low frequency aircraft instability involving control surface deflections.

II. Definition of Terms

Arrow-head wing with supersonic edges. Any one of the series of wings, which may be considered to be obtained from the delta or triangular wing with supersonic edges, by cutting the afterportion of the delta wing in such a manner that the flow normal to the resulting trailing edges remains supersonic, is called an arrow-head wing with supersonic edges. Several sketches of arrow-head wings are shown in figure 3.

Supersonic edge. Any edge, such as a leading or trailing edge, immersed in a flow whose component normal to this edge is supersonic is called a supersonic edge.

Purely supersonic. This phrase is used to characterize the boundary-value problem pertaining to a wing with supersonic edges. It is also used in describing the flow over a wing whose upper and lower surfaces may be regarded as acting independently of one another.

Section force or moment. The section force or moment per unit span that acts on a wing section taken parallel to the free stream direction.

III. Thesis Organization

In chapter II a brief review of some previous related studies is given. The text of the thesis is then presented in which the theoretical study is made under appropriate chapter divisions. The first part of the

text covers the formulation, discussion, and solution of the purely supersonic boundary-value problem for the velocity potential. The latter part gives the derivation of the air forces from the velocity potential and some discussion of these air forces.

IV. Symbols

The analysis to follow contains a wealth of symbols and a table summarizing these is now given. Although most of the symbols will be defined when introduced in the text, the following summary may prove useful.

ϕ	disturbance velocity potential
x,y,z	rectangular cartesian coordinates
U	velocity of main stream
∇	in rectangular coordinates, $\nabla = \underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}$
a	velocity of sound in main stream
t	time
w(x,y,t)	normal velocity at surface of wing, at point (x,y)
Z_m	function defining vertical displacement of a point (x,y) of the wing
p	disturbance pressure acting on wing surface
ρ	air density in main stream
h	vertical displacement of axis of rotation; $h = h_o e^{i\omega t}$
α	angle of attack; $\alpha = \alpha_o e^{i\omega t}$
$\dot{h}, \dot{\alpha}$	time derivatives of h and α respectively

ω	circular frequency of oscillation
M	freestream Mach number; $M = U/a$
β	$\sqrt{M^2 - 1}$
ξ, η	rectangular cartesian coordinates used to represent location of sources in xy-plane
$\bar{\omega}$	$M^2\omega/U\beta^2$
a_n, b_m	functions of $\bar{\omega}$, x, and M defined in Eq. 4-10
λ	slope of leading edge of wing as shown in Fig. 1
$\mathcal{Q}$	$\lim_{\beta\lambda \to 1} \phi$
u, v	characteristic coordinates; $u = \frac{M}{2\beta}(\xi - \beta\eta)$, $v = \frac{M}{2\beta}(\xi + \beta\eta)$
γ	$\frac{1 - \beta\lambda}{1 + \beta\lambda}$
σ	$\beta^2\lambda^2 - 1$
b	semi-chord of midspan wing section
μ, ν, μ_o	non-dimensional coordinates; $\mu = \frac{x}{2b}$, $\nu = \frac{y}{2b}$, $\mu_o = \frac{x_o}{2b}$
k	reduced frequency; $k = \frac{b\omega}{U}$
Δp	local surface pressure difference
P	section force, positive downward
M_a	section moment about axis of rotation, $x = x_o$, positive leading edge up
L_i, M_i	components of section force and moment coefficients, respectively, defined in Eqs. 5-12; $i = 1,2,3,4$
$\bar{P}$	total force on delta wing, positive downward

$\bar{M}_a$ total moment on delta wing about axis of rotation, $x = x_o$, positive leading edge up

$\bar{L}_i$, $\bar{M}_i$ components of total force and moment coefficients, respectively, defined in Eqs. 5-15; $i = 1,2,3,4$

E_{mn}, F_{mn}^{l}, G_{mn}, H_{mn}^{l} functions of μ_1, β, and ν defined in Eqs. 5-13

$\alpha_i, \beta_i, \gamma_i, \delta_i$; functions of M, λ , σ defined in Eqs. 5-14

l distance to point at which extended trailing edge of configuration (d), Fig. 3, intersects x-axis

s semi span of wing

CHAPTER II

HISTORICAL BACKGROUND

This thesis comprises a theoretical study of the aerodynamic forces on a thin, oscillating, triangular or related wing with supersonic edges. The treatment is based on solutions of the linearized differential equation for three dimensional supersonic flow. Therefore the wing in question is assumed to produce only small disturbances in the fluid, as it moves forward at a constant supersonic speed.

In 1947 Garrick and Rubinow[1] considered the problem of finding the air forces on any oscillating thin wing with supersonic edges by means of linearized theory. They are responsible for classifying, as "purely supersonic," the boundary-value problem for the velocity potential for this type of wing. Their study showed that the purely supersonic boundary-value problem is satisfied by a simple distribution of sources, with local strength proportional to the local prescribed normal velocity at the wing surface. An earlier study

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[1] I. E. Garrick and S. I. Rubinow, "Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream." NACA Report 872, 1947, 14 pp.

by A. E. Puckett[2] led to the same conclusion, but only with regard to the non-oscillating or steady wing with supersonic edges. Garrick and Rubinow did not explicitly treat the oscillating triangular wing, but their general solution[3] must be regarded as giving the velocity potential for this wing in integral form. The purpose of this thesis is to extend the study of Garrick and Rubinow by evaluating their integral-form solution for the particular case of an oscillating arrow-head wing.

There are several other papers closely associated with the subject of this thesis. Chang[4] and Li[5] in considering oscillating, finite, swept wings used an integral form for the velocity potential quite similar to that for the oscillating arrow-head wing. John W. Miles[6], more

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[2]Allen E. Puckett, "Supersonic Wave Drag of Thin Airfoils." Journal of the Aeronautical Sciences, Vol. 13, No. 9, 1946, pp. 475-84.

[3]Garrick and Rubinow, op. cit., p. 6.

[4]Chieh-Chien Chang, "The Aerodynamic Behavior of a Harmonically Oscillating Finite Sweptback Wing in Supersonic Flow," The Johns Hopkins University, School of Engineering, Department of Aeronautics, Nov. 1949.

[5]Ting-Yi Li, Periodic Supersonic Motions of a Thin Wing of Finite Span, Thesis, California Institute of Technology, 1950.

[6]John W. Miles, "On Harmonic Motion of Wide Delta Airfoils at Supersonic Speeds," TM RRB-36; U. S. Naval Ordnance Test Station, Inyokern, California, Feb. 9, 1950. 39 pp.

directly, obtained the integrated form for the velocity potential for the particular oscillating arrow-head wing with supersonic edges known as the "wide" delta wing. He used the velocity potential to find the total forces and moments on this wing. Also of interest is a paper by Brown and Adams[7]. They obtain, among other things, the pressure distribution on a pitching triangular wing with supersonic edges. If the velocity potential for pitching obtained in this thesis is reduced to terms containing only the first order of the frequency of oscillation, and is then used to find the pressure to only the first order of the frequency, the aforementioned result given by Brown and Adams is reproduced.

This thesis will effectively include the total forces and moments for the wide delta given by Miles, and in addition will give the section forces and moments not only for the wide delta but also for the more general type wing referred to herein as the arrow-head wing. The section rather than the total forces and moments are desirable, for example, in a strip flutter analysis that includes the wing flutter mode shape.

Expansion method of solution. The velocity potential for the triangular wing does not appear to be obtainable

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[7]Clinton E. Brown and Mac C. Adams, "Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds," NACA TN 1566, 1948. 29 pp.

in closed form in terms of elementary functions. The solution of the problem therefore is approximated by various types of expansion. Thus Miles[8] expanded the velocity potential in a series of Bessel functions, whereas Watkins[9], in the treatment of the rectangular wing, employed an expansion in powers of the frequency of oscillation. The latter method is utilized in this thesis.

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[8]Miles, op. cit.

[9]Charles E. Watkins, "Effect of Aspect Ratio on the Air Forces and Moments of Harmonically Oscillating Thin Rectangular Wings in Supersonic Potential Flow," NACA TN 2064, 1950. 52 pp.

CHAPTER III

FORMULATION OF PROBLEM

Consider uniform supersonic flow past a thin flat "arrow-head" type wing as shown in figure 1 (with its leading edges outside the Mach cone generated by the nose of the wing). The wing is referred to a coordinate system fixed in space and is assumed to be creating small disturbances in the main stream. Then if the undisturbed stream velocity is in the direction of the positive x axis and is of magnitude U, the equation[1] satisfied by the disturbance velocity potential for the wing is

$$\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\phi = \nabla^2\phi \qquad (3\text{-}1)$$

The wing is assumed to be performing oscillations of small amplitude about its own undisturbed position, the plane Z = 0. The boundary condition to be satisfied at the wing surface is then

$$\left(\frac{\partial\phi}{\partial z}\right)_{z\to 0} = w(x,y,t) = U\frac{\partial Z_m}{\partial x} + \frac{\partial Z_m}{\partial t} \qquad (3\text{-}2)$$

where Z_m is the vertical displacement of a point (x,y) of the wing. Note that this boundary condition is evaluated, in accordance with small disturbance linearized theory, at the xy projection of the wing rather than at the wing itself. There are two additional conditions

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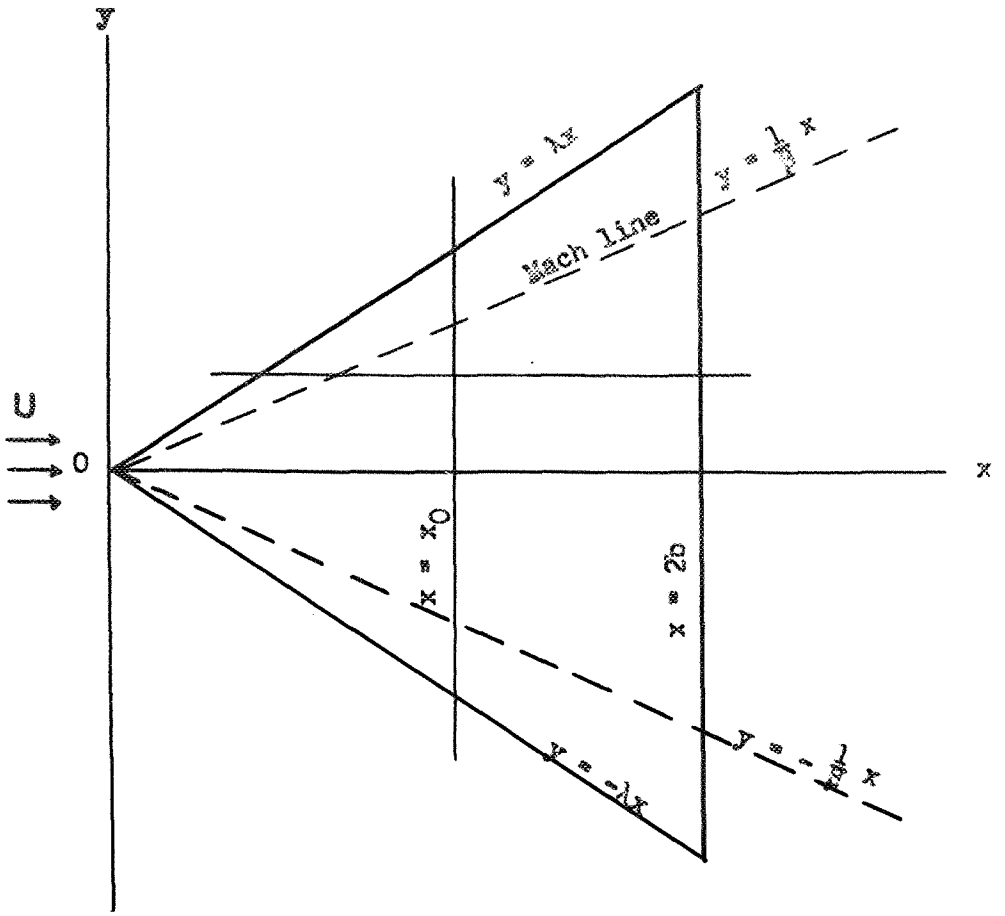
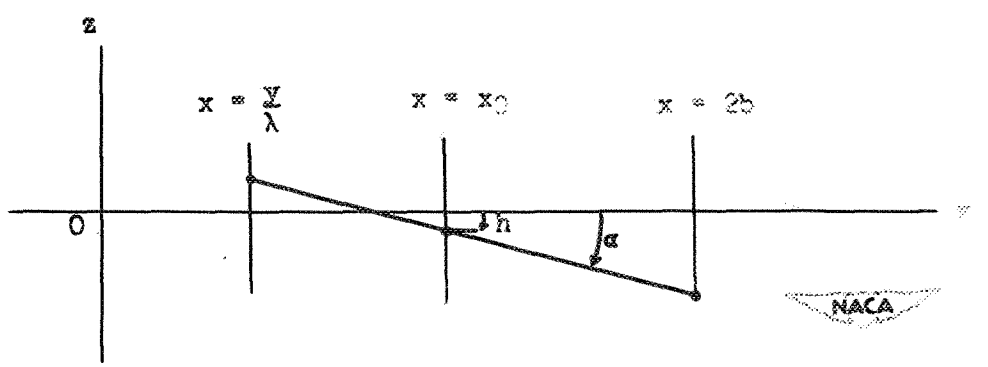
[1] Garrick and Rubinow, op. cit., p. 3.



(a) Projection of planform on xy-plane



(b) Section y

Figure 1.- Sketch illustrating coordinate system and the two degrees of freedom [illegible].

to which the velocity potential is subject. The first of these requires that the sources of disturbance, used in constructing the velocity potential, be felt only in their respective Mach cones. The second is a realization of the fact that only the wing can sustain a pressure difference. Each of these conditions is implicitly satisfied in the solution given in the subsequent text. Therefore, since Eqs. 3-1 and 3-2 are the only explicit mathematical statements involved in obtaining the velocity potential, they constitute the boundary-value problem for the velocity potential. Once the velocity potential ϕ has been obtained the pressure on either wing surface may be found by means of a linearized form of Bernoulli's equation,

$$p = -\rho\left(\frac{\partial \phi}{\partial t} + U\frac{\partial \phi}{\partial x}\right) \tag{3-3}$$

For the present problem of a wing performing small harmonic torsional oscillations of amplitude α_o about some spanwise axis x_o and small harmonic vertical translations of amplitude h_o (see Fig. 1b), the equation for Z_m is

$$Z_m = h + (x - x_o)\alpha = \left[h_o + (x - x_o)\alpha_o\right] e^{i\omega t} \tag{3-4}$$

where ω is the frequency of oscillation. Substitution of Eq. 3-4 into Eq. 3-2 gives

$$w(x, y, t) = \dot{h} + U\alpha + (x - x_o)\dot{\alpha} \tag{3-5}$$

For convenience, Eqs. 3-4 and 3-5 are expressed in terms of complex quantities. The use of such quantities allows a vectorial representation of the wing displacements a and h, and the forces and moments that are found in Chapter V. In a convenient manner there is introduced a means for obtaining either graphically or analytically any phase angle that might exist between a wing displacement and its associated force or moment. Furthermore a complex form for the air forces and moments is usually desired for flutter considerations. In a flutter analysis the use of complex quantities leads to a frequency equation containing real and imaginary parts, each of which must be zero individually. In this manner two equations are obtained in terms of two unknowns: the flutter speed U and the flutter frequency ω. The use of complex quantities in place of real or physical quantities is permissible because of the linearity of the equations being considered.

Since Eq. 3-1 is a linear differential equation, any number of individual solutions to it may be superimposed to obtain a more general solution. It is convenient on the basis of this to consider the velocity potential satisfying Eqs. 3-1 and 3-5 as the sum of three potentials, each being associated with one of the terms on the right hand side of Eq. 3-5. Thus the potential will be obtained in the form

$$\phi = \phi_h + \phi_\alpha + \phi_{\dot{\alpha}} \qquad (3\text{-}6)$$

This thesis is concerned only with the lift on a thin arrow-head wing with supersonic edges. Thickness effects are considered negligible. The wing, although oscillating with small amplitude, is assumed to be effectively in the plane Z = 0 at any time t. On the basis of these assumptions the normal velocity at any point (x,y) of either the top or bottom surface of the wing is approximately the same in magnitude and direction, and consequently the velocity potential for lift is antisymmetric with respect to the plane Z = 0. The velocity potential ϕ, given in Eq. 3-6 and developed more fully in the next chapter, applies to only one surface of the wing. The top surface is chosen for the analysis and because the antisymmetrical potential is simply opposite in sign on the bottom surface, the pressure difference between the upper and lower wing surfaces is obtained by multiplying the right hand side of Eq. 3-3 by 2 and using the ϕ obtained for the top surface.

CHAPTER VI

DERIVATION OF VELOCITY POTENTIAL

In this chapter the derivation of the velocity potential for the arrow-head wing with supersonic edges is given. First, however, it is necessary to obtain various forms for the elementary source solution of Eq. 3-1 that are useful in the derivation. The source solution is used to obtain the integral form of the velocity potential for the wing.

I. Elementary Source Solution

An elementary solution[1] of Eq. 3-1, that may be interpreted as a source of disturbance in a supersonic main stream of velocity U (see Figs. 1 and 2 for coordinate system used), is

$$\phi = -\frac{w(\xi,\eta)}{2\pi}\,\frac{w(t-\tau_1)+w(t-\tau_2)}{R} \qquad (4\text{-}1)$$

where $-w(\xi,\eta)/2\pi$ is the space variation of source strength, w(t) is the time variation of source strength and

$$\tau_{1,2} = \frac{M(x-\xi) \mp R}{a\beta^2}$$

$$R = \sqrt{(x-\xi)^2 - \beta^2[(y-\eta)^2+z^2]}$$

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[1] Garrick and Rubinow, op. cit.

In a supersonic main stream, a source of disturbance influences only the region of flow contained in the downstream opening Mach cone. Eq. 4-1 may be made to represent such a situation by restricting R to positive real values. Thus, Eq. 4-1 is assumed to apply to the region of flow contained in the cone whose vertex is at (ξ, η) and whose defining equation is

$$(y-\eta)^2+z^2 = \frac{1}{\beta^2}(x-\xi)^2$$

For the particular case of harmonic variations of the source strength given by (using complex notation as in Chapt. III)

$$w(t) = e^{i\omega t}$$

there is obtained from Eq. 4-1 the following relation:

$$\begin{aligned}\phi &= -\frac{w(\xi,\eta)}{2\pi}\frac{e^{i\omega t}}{R}\left(e^{-i\omega\tau_1}+e^{-i\omega\tau_2}\right)\\ &= -\frac{w(\xi,\eta)}{\pi}\frac{e^{i\omega t}}{R}e^{-i\omega\frac{\tau_1+\tau_2}{2}}\cos\left(\omega\frac{\tau_2-\tau_1}{2}\right)\\ &= -\frac{w(\xi,\eta)}{\pi}\frac{e^{i\omega t}}{R}e^{-i(x-\xi)\bar{\omega}}\cos\left(\bar{\omega}\frac{R}{M}\right)\end{aligned} \qquad (4\text{-}2)$$

where

$$\bar{\omega} = \frac{\omega M}{a\beta^2} = \frac{M^2\omega}{U\beta^2}$$

A form of Eq. 4-2 that will be of use later involves the Maclaurin's series expansion with respect to $\bar{\omega}$ of the portions of the equation that involve $\bar{\omega}$. The necessary expansions are

$$e^{-i(x-\xi)\bar{\omega}} = \sum_{m=0}^{\infty} (-i)^m \frac{(x-\xi)^m \bar{\omega}^m}{m!}$$

$$\cos\left(\bar{\omega}\frac{R}{M}\right) = \sum_{n=0}^{\infty} (-1)^n \left(\frac{R}{M}\right)^{2n} \frac{\bar{\omega}^{2n}}{(2n)!}$$

Thus, Eq. 4-2 may be expressed as

$$\phi = -\frac{w(\xi,\eta)}{\pi}\frac{e^{i\omega t}}{R}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(-1)^n(-i)^m\left(\frac{R}{M}\right)^{2n}\frac{(x-\xi)^m}{m!\,(2n)!}\bar{\omega}^{m+2n}$$

or rearranged to yield

$$\phi = -\frac{w(\xi,\eta)\,e^{i\omega t}}{\pi R}\sum_{m=0}^{\infty}\bar{\omega}^m\sum_{n=0}^{\left[\frac{m}{2}\right]}\frac{(-i)^m(x-\xi)^{m-2n}}{(2n)!\,(m-2n)!}\left(\frac{R}{M}\right)^{2n} \qquad (4\text{-}3)$$

where $\left[\frac{m}{2}\right]$ denotes the integral part of $\frac{m}{2}$.

Since Eq. 3-1 is a linear differential equation, a more general solution to it may be constructed by the superposition of potentials of the types given in Eqs. 4-1, 4-2, or 4-3. Such superpositions will be used in the next section of this chapter.

II. Equations for Velocity Potential

Integral form. The boundary-value problem for the arrow-head wing with supersonic edges, discussed in Chapt. III, is similar to that for the wing of infinite span treated by Garrick and Rubinow[2]. The problem in the reference paper is shown to be satisfied by a distribution or superposition of sources over the upper wing surface, of the type given in Eq. 4-1. Thus the velocity potential for the arrow-head wing is assumed to be

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[2]Garrick and Rubinow, op. cit., pp. 6-7

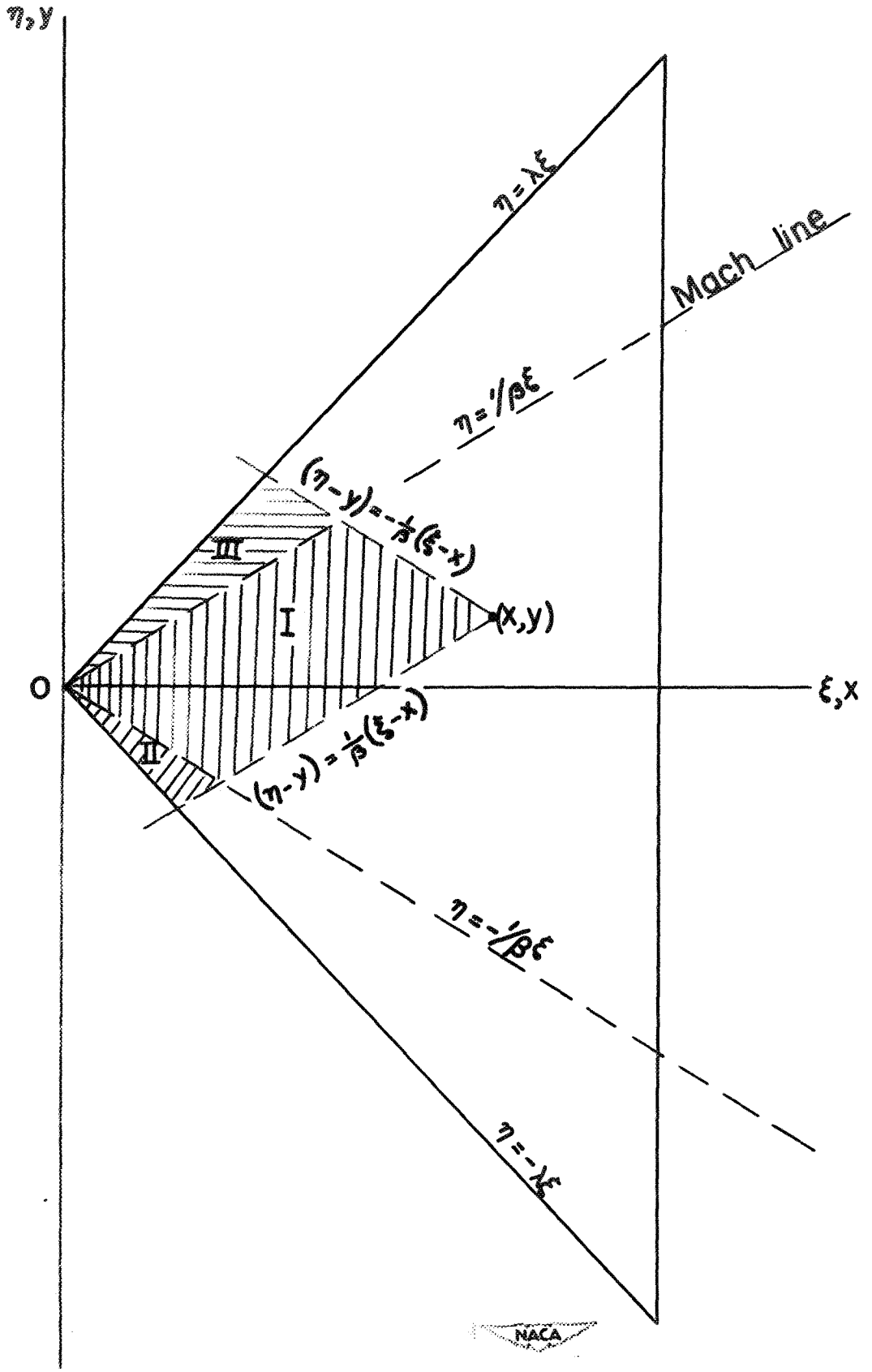

Figure 2.- Sketch illustrating the regions of integration for the velocity potential.

$$\phi(x,y,+o,t) = -\frac{1}{2\pi}\iint_S w(\xi,\eta)\frac{w(t-\tau_1)+w(t-\tau_2)}{R}\,d\eta\,d\xi \qquad (4\text{-}4)$$

where $w(\xi,\eta)$ now represents the surface variation of source strength that must be evaluated in accordance with the individual terms of Eq. 3-5 and

$$R = \sqrt{(x-\xi)^2-\beta^2(y-\eta)^2}$$

The region of integration S, in Eq. 4-1, is the part of the wing (in $\xi\eta$-plane) cut out by the reflected Mach cone opening upstream of the point (x,y, +o) as shown by the shaded area in Fig. 2. This may be seen from the fact that only those sources of disturbance whose after-cones include the point (x,y, +o), will have any effect on this point.

For the present problem of the wing performing small harmonic torsional and vertical oscillations, the superposition of sources of the type given in Eq. 4-2 is appropriate. Thus Eq. 4-4 takes the form

$$\phi(x,y,+o,t) = -\frac{e^{i\omega t}}{\pi}\iint_S \frac{w(\xi,\eta)}{R}\,e^{-i(x-\xi)\bar{\omega}}\cos\left(\bar{\omega}\frac{R}{M}\right)d\eta\,d\xi \qquad (4\text{-}5)$$

where the values of $w(\xi,\eta)$ for the harmonic case from Eq. 3-5 are:

for h $\qquad w(\xi,\eta) = i\omega h_o = \frac{iU\beta^2\bar{\omega}}{M^2}h_o$

for $U\alpha$ $\qquad w(\xi,\eta) = U\alpha_o$

for $\dot{\alpha}(X-X_o)$ $\qquad w(\xi,\eta) = \frac{iU\beta^2\bar{\omega}}{M^2}(\xi-X_o)\alpha_o$

For certain portions of the region of the wing (shaded in Fig. 2) to which Eq. 4-5 applies, it is possible to carry out the integration as in the case of the infinite wing[3]. However, the entire potential ϕ, as written in Eq. 4-5, does not appear to be obtainable in terms of known functions. Previous mention of this point was made in Chapt. II. In order to give consistent treatment to the entire region of integration, the integrand of Eq. 4-5 may be expanded in a Maclaurin's series with respect to $\bar{\omega}$, with the result that each term of the expansion can be readily integrated. Thus by substituting Eq. 4-3 in Eq. 4-5 there is obtained

$$\phi(x,y,+o,t) = -\frac{e^{i\omega t}}{\pi}\iint_S \frac{w(\xi,\eta)}{R}\sum_{m=o}^{\infty}\bar{\omega}^m\sum_{n=o}^{\left[\frac{m}{2}\right]}\frac{(-i)^m(x-\xi)^{m-2n}}{(2n)!\,(m-2n)!}\left(\frac{R}{M}\right)^{2n}d\eta\,d\xi \qquad (4\text{-}6)$$

Because Eq. 4-3 satisfies Eq. 3-1, Eq. 4-6 does also by virtue of the principal of superposition that applies to linear differential equations.

Satisfaction of boundary condition. It remains to be shown that Eq. 4-6 satisfies Eq. 3-5 and by so doing completely satisfies the boundary-value problem discussed in Chapt. III. To show this it is only necessary to consider Eq. 4-6, with $z \neq 0$, expressed to the second power of $\bar{\omega}$. Thus the potential at a point P above the wing is supposed to be given by

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[3]Garrick and Rubinow, loc. cit.

$$\phi_P = -\frac{e^{i\omega t}}{\pi}\iint_S w(\xi,\eta)\left[a_0\frac{1}{R}+a_1\frac{\xi}{R}+a_2\frac{\xi^2}{R}+b_0 R\right]d\eta d\xi \qquad (4\text{-}7)$$

where

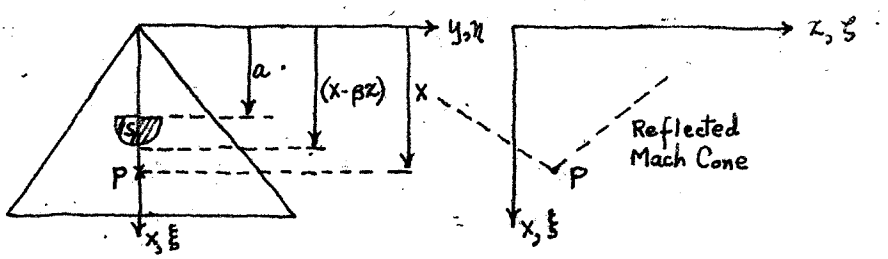
$$a_0 = 1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2}x^2$$

$$a_1 = i\bar{\omega} + \bar{\omega}^2 x$$

$$a_2 = -\frac{\bar{\omega}^2}{2}$$

$$b_0 = -\frac{\bar{\omega}^2}{2M^2}$$

For convenience, the point P will be chosen as shown in the following sketch, and only an element of the surface of integration will be considered:



Over the elemental area the distribution $w(\xi,\eta)$ may be considered constant by virtue of the mean value theorem for integrals. Thus Eq. 4-7 takes the form

$$\phi_P = -\frac{w(\xi,\eta)e^{i\omega t}}{\pi}\int_a^{x-\beta z}\int_{-\eta_1}^{\eta_1}\left[a_0\frac{1}{R}+a_1\frac{\xi}{R}+a_2\frac{\xi^2}{R}+b_0 R\right]d\eta d\xi \qquad (4\text{-}8)$$

where η_1 is given by the relation

$$\beta^2\left[(y-\eta_1)^2+z^2\right] = (x-\xi_1)^2$$

Now then it can be shown that

$$\lim_{z\to 0} \frac{\partial}{\partial z} \int_a^{x-\beta z} \int_{-\eta_1}^{\eta_1} \frac{\xi^n}{R}\, d\eta\, d\xi = -\pi x^n \qquad n \geq 0$$

$$\lim_{z\to 0} \frac{\partial}{\partial z} \int_a^{x-\beta z} \int_{-\eta_1}^{\eta_1} \xi^n R^{2m+1}\, d\eta\, d\xi = 0 \qquad m, n \geq 0$$

Therefore from Eq. 4-8

$$\left(\frac{\partial \phi_p}{\partial z}\right)_{z\to 0} = -\frac{w(\xi,\eta)e^{i\omega t}}{\pi}\left[-\pi\left(1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2}x^2 + i\bar{\omega}x + \bar{\omega}^2x^2 - \frac{\bar{\omega}^2}{2}x^2 + 0\cdots\right)\right]$$

$$= w(\xi,\eta)\, e^{i\omega t} \tag{4-9}$$

Eqs. 4-9 and 3-5 are identical. Since the terms in Eq. 4-9 associated with 1/R cancel identically and the terms associated with R^{2m+1} are zero, Eq. 3-5 is satisfied regardless of the order of $\bar{\omega}$ considered. Therefore, Eq. 4-6 satisfies the boundary-value problem under consideration. The preceding argument is similar to that employed by Puckett[4] for the steady case.

These foregoing considerations show that the boundary-value problem under consideration is satisfied by a surface distribution of sources, with local strength proportional to the local prescribed normal velocity of the wing surface. In addition, the velocity potential Eq. 4-6, for the upper surface of the wing, gives the proper normal velocity distribution only over this

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[4]Puckett, op. cit. p. 476

upper surface. To complete the physical picture, a corresponding normal velocity distribution is required over the lower surface of the wing. This requirement is met by simply changing the sign of Eq. 4-6, and considering the result as representing the lower surface of the wing. This consideration is automatically accounted for by multiplying Eq. 3-3 by 2 in obtaining the pressure difference between the two surfaces, as was previously mentioned in Chapter III.

Integrated form. The velocity potential to the third power of $\bar{\omega}$ is considered sufficient for a large number of practical applications. Thus Eq. 4-6 to the third power of $\bar{\omega}$, neglecting temporarily any $\bar{\omega}$ that may come from $w(\xi,\eta,t)$, is

$$\phi(x,y,+0,t) = -\frac{e^{i\omega t}}{\pi}\iint_S w(\xi,\eta)\left[a_0\frac{1}{R}+a_1\frac{\xi}{R}+a_2\frac{\xi^2}{R}+a_3\frac{\xi^3}{R}+b_0R+b_1\xi R\right]d\xi d\eta \qquad (4\text{-}10)$$

where

$$a_0 = 1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2}x^2 + i\frac{\bar{\omega}^3}{6}x^3$$
$$a_1 = i\bar{\omega} + \bar{\omega}^2 x - i\frac{\bar{\omega}^3}{2}x^2$$
$$a_2 = -\frac{\bar{\omega}^2}{2} + i\frac{\bar{\omega}^3}{2}x$$
$$a_3 = -i\frac{\bar{\omega}^3}{6}$$
$$b_0 = -\frac{\bar{\omega}^2}{2M^2} + i\frac{\bar{\omega}^3}{2M^2}x$$
$$b_1 = -i\frac{\bar{\omega}^3}{2M^2}$$

The question may be raised at this point as to whether or not Eq. 4-10 represents a solution of the boundary-value problem under discussion. With regard to this

question, it will be shown next that, when all the terms involving $\bar{\omega}$ up to a given power are taken into account, the differential equation, Eq. 3-1, is satisfied to this power. The boundary condition of tangential flow, Eq. 3-5, was previously shown to be satisfied exactly regardless of the order of $\bar{\omega}$ considered.

Write Eq. 4-10 in the extended form

$$\phi(x,y,+o,t) = -\frac{e^{i\omega t}}{\pi}\iint_S w(\xi,\eta)\left[\left(a_o\frac{1}{R}+b_oR+\cdots\right)\right.$$

$$\left.+\xi i\bar{\omega}\left(a_o\frac{1}{R}+\cdots\right)+\cdots\right]d\eta d\xi \qquad (4\text{-}11)$$

Since Eq. 3-1 is independent of ξ, the only portion of Eq. 4-11 that enters into the satisfaction of Eq. 3-1 is

$$\phi_1 = \left(a_o\frac{1}{R}+b_oR+\cdots\right) \qquad (4\text{-}12)$$

If it is assumed that the oscillatory motion is slow enough to be adequately represented by $\bar{\omega}$ to the first power then Eq. 4-12 becomes

$$\phi_1 = (1-i\bar{\omega}x)\frac{1}{R} \qquad (4\text{-}13)$$

Upon substituting Eq. 4-13 into Eq. 3-1 there are obtained the following residual terms:

$$-\frac{\bar{\omega}^2\beta^4}{M^2R}+2\beta^2\bar{\omega}^2\left(\frac{1}{R}-\frac{x^2}{R^3}\right)+\frac{i\beta^4\bar{\omega}^3}{M^2}\frac{x}{R}$$

Since $\bar{\omega}^n$, where $n > 1$, was considered negligibly small, each of the above terms is approximately zero.

Thus the statement following Eq. 4-10, that the differential equation is satisfied to the power of ϖ considered in Eq. 4-10, is verified.

Putting the values of $w(\xi,\eta)$ given in Eq. 4-5 into Eq. 4-10 and performing the indicated surface integration (over the shaded region in figure 2) yields the following forms for the velocity potentials in Eq. 3-6:

$$\begin{aligned}
\phi_h &= -\frac{h}{\pi}\left[A\sqrt{x^2-\beta^2y^2} + B_1 \cos^{-1}\frac{x+\beta^2\lambda y}{\beta(\lambda x+y)} + B_2 \cos^{-1}\frac{x-\beta^2\lambda y}{\beta(\lambda x-y)}\right] \\
&= -\frac{h}{\pi} F_1(x,y) \\
\phi_\alpha &= -\frac{U\alpha}{\pi} F_1(x,y) \qquad (4\text{-}14) \\
\phi_{\dot\alpha} &= -\frac{\dot\alpha}{\pi}\left[C\sqrt{x^2-\beta^2y^2} + D_1 \cos^{-1}\frac{x+\beta^2\lambda y}{\beta(\lambda x+y)} + D_2 \cos^{-1}\frac{x-\beta^2\lambda y}{\beta(\lambda x-y)} - x_0 F_1(x,y)\right]
\end{aligned}$$

The coefficients A, B_1, B_2, C, D_1, and D_2 are given in part III of this chapter. In performing the integrations that led to the results represented by Eq. 4-14, the use of characteristic coordinates (the coordinate lines are the Mach lines in Fig. 2) was found convenient. Reasons for this will be given in the next section, where a brief account of the derivation of Eq. 4-14 is presented.

In the limiting case where the Mach lines coincide with the leading edges of the wing, that is when $\beta\lambda = 1$, Eqs. 4-14 take the form:

$$\begin{aligned}
\mathcal{U}_h &= -\frac{h}{\pi} A_1 \sqrt{x^2-\beta^2y^2} \\
\mathcal{U}_\alpha &= -\frac{U\alpha}{\pi} A_1 \sqrt{x^2-\beta^2y^2} \qquad (4\text{-}15) \\
\mathcal{U}_{\dot\alpha} &= -\frac{\dot\alpha}{\pi}(C_1 - x_0 A_1)\sqrt{x^2-\beta^2y^2}
\end{aligned}$$

The coefficients A_1 and C_1 and the derivation of Eq. 4-15 are given in part III.

III. Coefficients of Velocity Potential Equations

A derivation of Eqs. 4-14 and 4-15 is given briefly here. The surface integral represented by Eq. 4-10 when written in terms of the characteristic coordinates uv becomes,

$$\phi = -\frac{e^{i\omega t}}{\pi}\left[\sum_{n=0}^{3}\frac{\beta^n}{M^{n+1}}\int_{v_1}^{v_2}\int_{u_1}^{u_2} w(u,v)\, b_n \frac{(u+v)^n\, du\, dv}{\sqrt{(u_o-u)(v_o-v)}}\right.$$

$$\left. + \sum_{m=0}^{1}\frac{4\beta^{m+2}}{M^{m+3}}\int_{v_1}^{v_2}\int_{u_1}^{u_2} w(u,v) b_m (u+v)^m \sqrt{(u_o-u)(v_o-v)}\, du\, dv\right] \quad (4\text{-}16)$$

where

$$u = \frac{M}{2\beta}(\xi - \beta\eta) \quad , \quad u_o = \frac{M}{2\beta}(x - \beta y)$$

$$v = \frac{M}{2\beta}(\xi + \beta\eta) \quad , \quad v_o = \frac{M}{2\beta}(x + \beta y)$$

The limits of integration in Eq. 4-16, from the characteristic coordinates previously defined and from Fig: 2, are:

Region	u_1	u_2	v_1	v_2
I	0	u_o	0	v_o
II	$\frac{1}{\gamma}v$	u_o	γu_o	0
III	γv	0	0	v_o

(4-17)

where

$$\gamma = \frac{1-\beta\lambda}{1+\beta\lambda}$$

Putting the appropriate values of $w(u,v)$ and the limits of integration given in Eq. 4-17 into Eq. 4-16 and performing the indicated integrations[5], and then reverting to xy coordinates, yields the potential in the form given in Eq. 4-14. By retaining only those terms associated with $\dot{h}$ or α that involve $\bar{\omega}$ up to the third power, the coefficients in Eq. 4-14 are found to be

$$A = i\frac{\lambda x\bar{\omega}}{\sigma} + \frac{\lambda x^2\bar{\omega}^2}{6M^2\sigma^2}\left[M^2(4\sigma+3)+\sigma\right] + \frac{\lambda\beta^2y^2\bar{\omega}^2}{6M^2\sigma^2}(3M^2+\sigma)$$
$$- i\frac{\lambda x^3\bar{\omega}^3}{72M^2\sigma^3}\left[M^2(18\sigma^2+31\sigma+15)+3\sigma(5\sigma+3)\right]$$
$$- i\frac{\lambda\beta^2xy^2\bar{\omega}^3}{72M^2\sigma^3}\left[M^2(38\sigma+45)+3\sigma(4\sigma+9)\right]$$

$$B_1(x,y) = \frac{1}{\sigma^{1/2}}(\lambda x+y) - i\frac{\lambda\beta^2\bar{\omega}}{2\sigma^{3/2}}(\lambda x+y)^2 - \frac{\beta^2\bar{\omega}^2}{12M^2\sigma^{5/2}}\left[M^2(2\sigma+3)+\sigma\right](\lambda x+y)^3$$
$$+ i\frac{\lambda\beta^4\bar{\omega}^3}{48M^2\sigma^{7/2}}\left[M^2(2\sigma+5)+3\sigma\right](\lambda x+y)^4$$

$$B_2(x,y) = B_1(x,-y)$$

$$C = \frac{\lambda x}{\sigma} - i\frac{\lambda x^2\bar{\omega}}{3\sigma^2}(\sigma+3) - i\frac{\lambda\beta^2y^2\bar{\omega}}{\sigma^2} - \frac{\lambda x^3\bar{\omega}^2}{24M^2\sigma^3}\left[M^2(2\sigma^2+19\sigma+15)+\sigma(\sigma+3)\right] \tag{4-18}$$
$$- \frac{\lambda\beta^2xy^2\bar{\omega}^2}{24M^2\sigma^3}\left[M^2(26\sigma+45)+9\sigma\right]$$

$$D_1(x,y) = -\frac{x}{\sigma^{3/2}}(\lambda x+y) - \frac{\lambda\beta^2y}{\sigma^{3/2}}(\lambda x+y) + \frac{\lambda\beta^2}{2\sigma^{3/2}}(\lambda x+y)^2 - i\frac{\lambda\beta^2x\bar{\omega}}{2\sigma^{3/2}}(\lambda x+y)^2$$
$$+ i\frac{\beta^2\bar{\omega}}{6\sigma^{5/2}}(2\sigma+3)(\lambda x+y)^3 + \frac{\beta^2x\bar{\omega}^2}{48M^2\sigma^{7/2}}\left[4M^2(4\sigma+5)-\sigma(\sigma-3)\right](\lambda x+y)^3$$
$$+ \frac{\lambda\beta^4y\bar{\omega}^2}{48M^2\sigma^{7/2}}\left[4M^2(2\sigma+5)+3\sigma\right](\lambda x+y)^3 - \frac{\lambda\beta^4\bar{\omega}^2}{48\sigma^{7/2}}(2\sigma+5)(\lambda x+y)^4$$

$$D_2(x,y) = D_1(x,-y)$$

---

[5]The integrations are not difficult but they are cumbersome. Therefore, no attempt has been made to give them here. The results have been checked several times and are felt to be correct as presented.

where

$$\sigma = \beta^2\lambda^2 - 1$$

In the case where the velocity components normal to the leading edges are sonic, or where the Mach lines coincide with these edges (that is, when $\beta\lambda = 1$, $\gamma = 0$, or $\sigma = 0$), it may be noted from Eqs. 4-17 and Fig. 2 that regions II and III no longer exist. Therefore, the potential in this case is determined by integration over region I only. The coefficients of Eqs. 4-15 are found to be

$$A_1 = \frac{2}{\beta}\left\{1 - i\frac{X\bar{\omega}}{3} - \frac{\bar{\omega}^2}{90M^2}\left[5(X^2-\beta^2y^2)+M^2(7X^2+2\beta^2y^2)\right]\right.$$
$$\left. + i\frac{X\bar{\omega}^3}{210M^2}\left[7(X^2-\beta^2y^2)+M^2(3X^2+2\beta^2y^2)\right]\right\} \qquad (4\text{-}19)$$

$$C_1 = \frac{2}{\beta}\left\{\frac{2X}{3} - i\frac{2\bar{\omega}}{45}(4X^2-\beta^2y^2) - \frac{X\bar{\omega}^2}{315M^2}\left[7(X^2-\beta^2y^2)\right.\right.$$
$$\left.\left. + M^2(11X^2-2\beta^2y^2)\right]\right\}$$

The use of characteristic coordinates in this section leads to very simple expressions for the limits of integration given in Eq. 4-17. In addition, their use simplifies the derivation of Eq. 4-19.

CHAPTER V

FORCES AND MOMENTS

I. Form of Force and Moment Equations

The preceding result for the velocity potential, Eq. 4-14, will now be used to obtain the forces and moments on any one of a series of wings generally referred to herein as "arrow-head" type. As stated in the introduction, the wings being considered are those that may be formed from the delta or triangular wing by cutting the afterportion of this wing in such a manner that the resulting trailing edges lie ahead of the Mach cones emanating from their foremost points. Sketches of several wing plan forms having the aforementioned characteristics are shown in Fig. 3.

Section forces and moments will be derived in general form, that is, in the form applicable to any of the aforementioned wings. These forces and moments are useful, for example, in a "strip-type" of flutter analysis. Total forces and moments will also be derived but only for the delta wing. The damping part of the total moment due to α_o will be used in a study of the possible loss of aerodynamic damping (loss will indicate that single degree torsional flutter is possible) of an oscillating delta wing which will infer the same possibility with regard to the other wings considered.

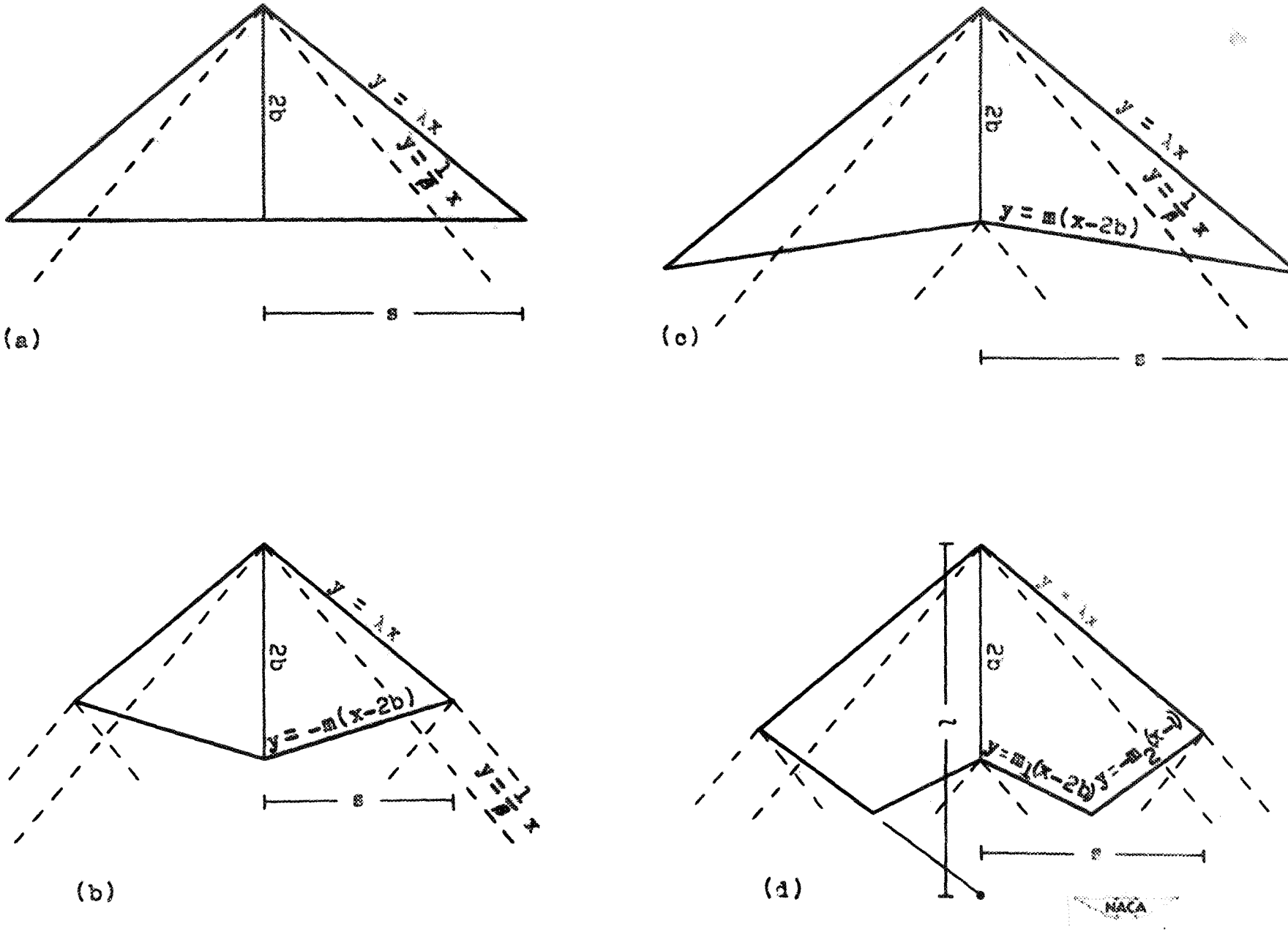

Figure 3.- Sketches illustrating different plan forms for which the force equations (19) and moment equations (20) apply.

In deriving the expressions for the section force and moment coefficients and the total force and moment coefficients it is convenient to employ the variables μ, ν, and μ_o as the non-dimensional quantities obtained by dividing the variables x, y, and x_o by the chord, 2b, of the midspan section of the wing, and to introduce the reduced frequency $k = \frac{b\omega}{U}$.

Section forces and moments. The local pressure difference between the upper and lower surfaces of the wing may be written by means of Eq. 3-3 as,

$$\Delta p = -2\rho\left(\frac{\partial \phi}{\partial t} + \frac{U}{2b}\frac{\partial \phi}{\partial \mu}\right) \tag{5-1}$$

The expression for the section force, positive downward, is therefore

$$P = -2b\int_{\nu/\lambda}^{\mu_1} \Delta p \, d\mu \tag{5-2}$$

and the section moment, positive nose-up, about the arbitrary axis of rotation $x = x_o$ is

$$M_a = -4b^2\int_{\nu/\lambda}^{\mu_1} (\mu-\mu_o)\,\Delta p \, d\mu \tag{5-3}$$

The limit of integration μ_1 in Eqs. 5-2 and 5-3 has the following values with respect to the different configurations shown in Fig. 3:

for configuration (a) $\mu_1 = 1$

for configuration (b) $\mu_1 = 1 - \frac{\nu}{m}$

for configuration (c) $\mu_1 = 1 + \frac{\nu}{m}$

for configuration (d)

$$\mu_1 = 1 + \frac{\nu}{m_1} \quad \text{for} \quad 0 \le \nu \le \frac{m_1 m_2}{m_1+m_2}\left(\frac{\ell}{2b}-1\right)$$

$$\mu_1 = \frac{\ell}{2b} - \frac{\nu}{m_2} \quad \text{for} \quad \frac{m_1 m_2}{m_1+m_2}\left(\frac{\ell}{2b}-1\right) \le \nu \le \frac{s}{2b} \tag{5-4}$$

After substituting Eq. 4-14 into Eqs. 5-2 and 5-3 and performing the indicated integrations the results can be written as

$$P = -4\rho b\, U^2 k^2 e^{i\omega t}\left[\frac{h_0}{b}(L_1+iL_2)+\alpha_0(L_3+iL_4)\right] \tag{5-5}$$

and

$$M_\alpha = -4\rho b^2 U^2 k^2 e^{i\omega t}\left[\frac{h_0}{b}(M_1+iM_2)+\alpha_0(M_3+iM_4)\right] \tag{5-6}$$

In Eqs. 5-5 and 5-6 the reduced frequency k, on which the L_i's and M_i's are dependent, is related to ω and $\bar{\omega}$ in the following manner

$$k = \frac{b\omega}{U} = \frac{b\beta^2}{M^2}\bar{\omega} \tag{5-7}$$

A brief account of the derivation of the force and moment coefficients L_i and M_i (where i = 1, 2, 3, 4) appearing in Eqs. 5-5 and 5-6 is given in part II of this chapter.

In Eqs. 5-5 and 5-6 the coefficients $(L_1 + iL_2)$, $(M_1 + iM_2)$ and $(L_3 + iL_4)$, $(M_3 + iM_4)$ are the section lift and moment coefficients associated with vertical translation and rotation respectively. The real

component of $(L_1 + iL_2)$, for example, is in phase with the displacement h_o and the imaginary part is 90^o out of phase with this displacement. Similar interpretations can be given to the remaining components. The imaginary part of each of the coefficients is proportional to the aerodynamic damping force or moment associated with the respective motion.

Total forces and moments for a delta wing. The expression for the total force, positive downward, on configuration (a) of Fig. 3 is

$$\bar{P} = -8b^2 \int_0^{\lambda} \int_{\nu/\lambda}^{1} \Delta\varphi \, d\mu \, d\nu \qquad (5\text{-}8)$$

The total moment, positive nose-up, about the arbitrary axis of rotation $x = x_o$ is

$$\bar{M}_\alpha = -16b^3 \int_0^{\lambda} \int_{\nu/\lambda}^{1} \Delta\varphi \, (x - x_o) \, d\mu \, d\nu \qquad (5\text{-}9)$$

After substituting Eqs. 4-14 into Eqs. 5-8 and 5-9 and performing the indicated integrations the results can be written as (using barred letters to designate total forces and moments)

$$\bar{P} = -8\rho b^2 U^2 k^2 e^{i\omega t} \left[\frac{h_o}{b} (\bar{L}_1 + i\bar{L}_2) + \alpha_o (\bar{L}_3 + i\bar{L}_4) \right] \qquad (5\text{-}10)$$

and

$$\bar{M}_\alpha = -8\rho b^3 U^2 k^2 e^{i\omega t} \left[\frac{h_o}{b} (\bar{M}_1 + i\bar{M}_2) + \alpha_o (\bar{M}_3 + i\bar{M}_4) \right] \qquad (5\text{-}11)$$

The derivation of the total force and moment coefficients $\bar{L}_i$ and $\bar{M}_i$ (where $i = 1, 2, 3, 4$) is also given in part II of this chapter.

II. Force and Moment Coefficients

Section force and moment coefficients. The coefficients L_i and M_i (where $i = 1,2,3,4$) are obtained by substituting Eq. 4-14 into Eqs. 5-2 and 5-3 and grouping the results in the form given in Eqs. 5-5 and 5-6. They therefore are defined as follows:

$$
\begin{aligned}
L_1 + iL_2 &= L_1' + iL_2' \\
L_3 + iL_4 &= L_3' + iL_4' - \left(\frac{i}{k} + 2\mu_0\right)(L_1' + iL_2') \\
M_1 + iM_2 &= M_1' + iM_2' - 2\mu_0(L_1' + iL_2') \\
M_3 + iM_4 &= M_3' + iM_4' - 2\mu_0(L_3' + iL_4') - \left(\frac{i}{k} + 2\mu_0\right)(M_1 + iM_2)
\end{aligned}
\qquad (5\text{-}12)
$$

where

$$
\begin{aligned}
L_1' + iL_2' &= -\frac{1}{2\pi}\left[2\int_{\nu/\lambda}^{\mu_1} F_1(2\mu, 2\nu)\, d\mu - \frac{i}{k} F_1(2\mu_1, 2\nu)\right] \\
L_3' + iL_4' &= -\frac{1}{2\pi}\left[2\int_{\nu/\lambda}^{\mu_1} F_2(2\mu, 2\nu)\, d\mu - \frac{i}{k} F_2(2\mu_1, 2\nu)\right] \\
M_1' + iM_2' &= -\frac{1}{2\pi}\left[4\int_{\nu/\lambda}^{\mu_1} F_1(2\mu, 2\nu)\,\mu\, d\mu - \frac{i}{k} 2\mu_1 F_1(2\mu_1, 2\nu)\right] \\
M_3' + iM_4' &= -\frac{1}{2\pi}\left[4\int_{\nu/\lambda}^{\mu_1} F_2(2\mu, 2\nu)\,\mu\, d\mu - \frac{i}{k} 2\mu_1 F_2(2\mu_1, 2\nu)\right]
\end{aligned}
$$

In Eqs. 5-12 the quantity $F_1(2\mu, 2\nu)$ is obtained from the quantity $F_1(x,y)$, defined in connection with Eq. 4-14, by replacing x by 2μ, y by 2ν and regarding the $\bar{\omega}$ associated with the coefficients A, B_1, and B_2 in $F_1(x,y)$ as $\bar{\omega}'$ where

$$\bar{\omega}' = \frac{M^2 k}{\beta^2}$$

The quantity $F_2(2\mu, 2\nu')$ is obtained in a similar manner from the expression

$$F_2(x,y) = C\sqrt{x^2-\beta^2y^2} + D_1 \cos^{-1}\frac{x+\beta^2\lambda y}{\beta(\lambda x+y)} + D_2 \cos^{-1}\frac{x-\beta^2\lambda y}{\beta(\lambda x-y)}$$

the right hand side of which is recognized as part of ϕ_α in Eqs. 4-14.

Consider now the following general definitions, particular forms of which are contained in L_i and M_i $(i = 1,2,3,4)$:

$$E_{mn} = 2^{m+n+1}\left[\nu^m \mu_1^n \sqrt{\mu_1^2-\beta^2\nu^2}\right]$$

$$F^{\ell}_{mn} = 2^{\ell+m+n}\left[(\beta^2\lambda\nu)^{\ell}\mu_1^m(\lambda\mu_1+\nu)^n \cos^{-1}\frac{\mu_1+\beta^2\lambda\nu}{\beta(\lambda\mu_1+\nu)} + (-\beta^2\lambda\nu)^{\ell}\mu_1^m(\lambda\mu_1-\nu)^n \cos^{-1}\frac{\mu_1-\beta^2\lambda\nu}{\beta(\lambda\mu_1-\nu)}\right]$$

$$G_{mn} = 2^{m+n+2}\left[\nu^m\int_{\nu/\lambda}^{\mu_1}\mu^n\sqrt{\mu^2-\beta^2\nu^2}\,d\mu\right] \qquad (5\text{-}13)$$

$$H^{\ell}_{mn} = 2^{\ell+m+n+1}\left[(\beta^2\lambda\nu)^{\ell}\int_{\nu/\lambda}^{\mu_1}\mu^m(\lambda\mu+\nu)^n \cos^{-1}\frac{\mu+\beta^2\lambda\nu}{\beta(\lambda\mu+\nu)}\,d\mu + (-\beta^2\lambda\nu)^{\ell}\int_{\nu/\lambda}^{\mu_1}\mu^m(\lambda\mu-\nu)^n \cos^{-1}\frac{\mu-\beta^2\lambda\nu}{\beta(\lambda\mu-\nu)}\,d\mu\right]$$

With the aid of Eqs. 5-13, the L_i' and M_i' $(i = 1,2,3,4)$ of Eqs. 5-12 may be written as

$$-2\pi L_1' = (\alpha_1 E_{01} + \beta_1 H_{01}^{\circ} + \beta_2 F_{02}^{\circ}) + (\alpha_2 G_{02} + \alpha_3 G_{20} + \alpha_4 E_{03} + \alpha_5 E_{21} + \beta_3 H_{03}^{\circ} + \beta_4 F_{04}^{\circ}) k^2$$

$$-2\pi L_2' = -\beta_1 F_{01}^{\circ} \frac{1}{k} + (\alpha_1 G_{01} - \alpha_2 E_{02} - \alpha_3 E_{20} + \beta_2 H_{02}^{\circ} - \beta_3 F_{03}^{\circ}) k$$

$$-2\pi L_3' = \gamma_1 G_{01} + \gamma_2 E_{02} + \gamma_3 E_{20} + \delta_1 H_{11}^{\circ} + \delta_2 H_{01}^{1} + \delta_3 H_{02}^{\circ} + \delta_4 F_{12}^{\circ} + \delta_5 F_{03}^{\circ}$$

$$-2\pi L_4' = -(\gamma_1 E_{01} + \delta_1 F_{11}^{\circ} + \delta_2 F_{01}' + \delta_3 F_{02}^{\circ}) \frac{1}{k}$$
$$+ (\gamma_2 G_{02} + \gamma_3 G_{20} - \gamma_4 E_{03} - \gamma_5 E_{21} + \delta_4 H_{12}^{\circ} + \delta_5 H_{03}^{\circ}$$
$$- \delta_6 F_{13}^{\circ} - \delta_7 F_{03}' - \delta_8 F_{04}^{\circ}) k$$

(5-14)

$$-2\pi M_1' = [\alpha_1 (E_{02} - G_{01}) + \beta_1 H_{11}^{\circ} + \beta_2 (F_{12}^{\circ} - H_{02}^{\circ})] + [\alpha_2 G_{03} + \alpha_3 G_{21} + \alpha_4 (E_{04} - G_{03})$$
$$+ \alpha_5 (E_{22} - G_{21}) + \beta_3 H_{13}^{\circ} + \beta_4 (F_{14}^{\circ} - H_{04}^{\circ})] k^2$$

$$-2\pi M_2' = \beta_1 (H_{01}^{\circ} - F_{11}^{\circ}) \frac{1}{k} + [\alpha_1 G_{02} + \alpha_2 (G_{02} - E_{03}) + \alpha_3 (G_{20} - E_{21}) + \beta_2 H_{12}^{\circ} + \beta_3 (H_{03}^{\circ} - F_{13}^{\circ})] k$$

$$-2\pi M_3' = \gamma_1 G_{02} + \gamma_2 (E_{03} - G_{02}) + \gamma_3 (E_{21} - G_{20}) + \delta_1 H_{21}^{\circ} + \delta_2 H_{11}' + \delta_3 H_{12} + \delta_4 (F_{22}^{\circ} - H_{12}^{\circ})$$
$$+ \delta_5 (F_{13}^{\circ} - H_{03}^{\circ})$$

$$-2\pi M_4' = [\gamma_1 (G_{01} - E_{02}) + \delta_1 (H_{11}^{\circ} - F_{21}^{\circ}) + \delta_2 (H_{01}' - F_{11}') + \delta_3 (H_{02}^{\circ} - F_{12}^{\circ})] \frac{1}{k}$$
$$+ [\gamma_2 G_{03} + \gamma_3 G_{21} + \gamma_4 (G_{03} - E_{04}) + \gamma_5 (G_{21} - E_{22}) + \delta_4 H_{22}^{\circ} + \delta_5 H_{13}^{\circ}$$
$$+ \delta_6 (H_{13}^{\circ} - F_{23}^{\circ}) + \delta_7 (H_{03}' - F_{13}') + \delta_8 (H_{04}^{\circ} - F_{14}^{\circ})] k$$

where

$$\alpha_1 = \frac{M^2 \lambda}{\beta^2 \sigma} \qquad \alpha_3 = \frac{M^2 \lambda}{6\beta^2 \sigma^2} (3M^2 + \sigma)$$

$$\alpha_2 = \frac{M^2 \lambda}{6\beta^4 \sigma^2} [M^2 (4\sigma + 3) + \sigma] \qquad \alpha_4 = -\frac{M^4 \lambda}{72\beta^6 \sigma^3} [M^2 (18\sigma^2 + 31\sigma + 13) + 3\sigma (5\sigma + 3)]$$

$$\alpha_5 = -\frac{M^4 \lambda}{72\beta^4 \sigma^3} [M^2 (38\sigma + 45) + 3\sigma (4\sigma + 9)]$$

$$\beta_1 = \frac{1}{\sigma^{1/2}} \qquad \beta_3 = -\frac{M^2}{12\beta^2\sigma^{5/2}}\left[M^2(2\sigma+3)+\sigma\right]$$

$$\beta_2 = -\frac{M^2\lambda}{2\sigma^{3/2}} \qquad \beta_4 = \frac{M^4\lambda}{48\beta^2\sigma^{7/2}}\left[M^2(2\sigma+5)+3\sigma\right]$$

$$\gamma_1 = \frac{\lambda}{\sigma} \qquad \gamma_3 = -\frac{M^2\lambda}{\sigma^2}$$

$$\gamma_2 = -\frac{M^2\lambda}{3\beta^2\sigma^2}(\sigma+3) \qquad \gamma_4 = -\frac{M^2\lambda}{24\beta^4\sigma^3}\left[M^2(2\sigma^2+19\sigma+15)+\sigma(\sigma+3)\right]$$

$$\gamma_5 = -\frac{M^2\lambda}{24\beta^2\sigma^3}\left[M^2(26\sigma+45)+9\sigma\right]$$

$$\delta_1 = -\frac{1}{\sigma^{3/2}} \qquad \delta_5 = \frac{M^2}{60\sigma^{5/2}}(2\sigma+3)$$

$$\delta_2 = \delta_1 \qquad \delta_6 = \frac{M^2}{48\beta^2\sigma^{7/2}}\left[4M^2(4\sigma+3)-\sigma(\sigma-3)\right]$$

$$\delta_3 = \frac{\beta^2\lambda}{2\sigma^{3/2}} \qquad \delta_7 = \frac{M^2}{48\beta^2\sigma^{7/2}}\left[4M^2(2\sigma+3)+3\sigma\right]$$

$$\delta_4 = -\frac{M^2\lambda}{2\sigma^{3/2}} \qquad \delta_8 = -\frac{M^4\lambda}{48\sigma^{7/2}}\left[2\sigma+3\right]$$

The coefficients in Eqs. 5-14, that is α_1, β_1, γ_1, and δ_1, originate respectively from the coefficients A, B_1 or B_2, C, and D_1 or D_2 defined in part III of Chapter IV. The quantities E_{mn} and F_{mn}^{l} contained in Eqs. 5-14 are easily determined from Eqs. 5-13. The quantities G_{mn} and H_{mn}^{l} however, are determined by integration, as indicated in Eqs. 5-13. The integral results[1] that are required in Eqs. 5-14 are:

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[1]The integrals again are not difficult to handle but are exceedingly cumbersome. Thus, as before, the details are omitted.

$$G_{01} = \frac{8}{3}\left(\mu_1^2 - \beta^2\nu^2\right)^{3/2}$$

$$G_{02} = 2\left[2\mu_1\left(\mu_1^2 - \beta^2\nu^2\right)^{3/2} + \beta^2\nu^2\mu_1\left(\mu_1^2 - \beta^2\nu^2\right)^{1/2} - \beta^4\nu^4\cosh^{-1}\frac{\mu_1}{\beta\nu}\right]$$

$$G_{03} = \frac{32}{15}\left(3\mu_1^2 + 2\beta^2\nu^2\right)\left(\mu_1^2 - \beta^2\nu^2\right)^{1/2}$$

$$G_{20} = 8\nu^2\left[\mu_1\left(\mu_1^2 - \beta^2\nu^2\right)^{1/2} - \beta^2\nu^2\cosh^{-1}\frac{\mu_1}{\beta\nu}\right]$$

$$G_{21} = \frac{32\nu^2}{3}\left(\mu_1^2 - \beta^2\nu^2\right)^{3/2}$$

$$H_{01}^{0} = \frac{1}{2\lambda}F_{02}^{0} - \frac{4\sigma^{1/2}}{\lambda}\nu^2\cosh^{-1}\frac{\mu_1}{\beta\nu}$$

$$H_{02}^{0} = \frac{1}{3\lambda}F_{03}^{0} - \frac{32\sigma^{1/2}}{3}\nu^2\sqrt{\mu_1^2 - \beta^2\nu^2}$$

$$H_{03}^{0} = \frac{1}{4\lambda}F_{04}^{0} - \frac{4\sigma^{1/2}}{\lambda}\left[3\mu_1\lambda^2\nu^2\sqrt{\mu_1^2 - \beta^2\nu^2} + \nu^4(3\sigma+5)\cosh^{-1}\frac{\mu_1}{\beta\nu}\right]$$

$$H_{04}^{0} = \frac{1}{5\lambda}F_{05}^{0} - \frac{256\sigma^{1/2}}{15\lambda}\left[\lambda\nu^4(2\sigma+5) + \lambda^3\mu_1^2\nu^2\right]\sqrt{\mu_1^2 - \beta^2\nu^2}$$

$$H_{11}^{0} = -\frac{1}{6\lambda^2}F_{03}^{0} + \frac{1}{2\lambda}F_{12}^{0} - \frac{8\sigma^{1/2}}{3\lambda}\nu^2\sqrt{\mu_1^2 - \beta^2\nu^2}$$

$$H_{12}^{0} = -\frac{1}{12\lambda^2}F_{04}^{0} + \frac{1}{3\lambda}F_{13}^{0} - \frac{4\sigma^{1/2}}{3\lambda^2}\left[5\lambda^2\nu^2\mu_1\sqrt{\mu_1^2 - \beta^2\nu^2} + \nu^4(5\sigma+3)\cosh^{-1}\frac{\mu_1}{\beta\nu}\right]$$

$$H_{13}^{0} = -\frac{1}{20\lambda^2}F_{05}^{0} + \frac{1}{4\lambda}F_{14}^{0} - \frac{16\sigma^{1/2}}{15\lambda^2}\left[(22\sigma+25)\lambda\nu^4 + 11\lambda^3\nu^2\mu_1^2\right]\sqrt{\mu_1^2 - \beta^2\nu^2}$$

$$H_{21}^{0} = \frac{1}{12\lambda^3}F_{04}^{0} - \frac{1}{8\lambda^2}F_{13}^{0} + \frac{1}{2\lambda}F_{22}^{0} - \frac{4\sigma^{1/2}}{3\lambda^2}\left[\lambda^2\nu^2\mu_1\sqrt{\mu_1^2 - \beta^2\nu^2} + \nu^4(\sigma+3)\cosh^{-1}\frac{\mu_1}{\beta\nu}\right.$$

$$H_{22}^{0} = \frac{1}{30\lambda^3}F_{05}^{0} - \frac{1}{6\lambda^2}F_{14}^{0} + \frac{1}{3\lambda}F_{23}^{0} - \frac{32\sigma^{1/2}}{15\lambda^3}\left[3\lambda^3\nu^2\mu_1^2 + \lambda\nu^4(6\sigma+5)\right]\sqrt{\mu_1^2 - \beta^2\nu^2}$$

$$H_{01}^{1} = \frac{1}{2\lambda}F_{02}^{1} - 8\sigma^{1/2}\beta^2\lambda\nu^2\sqrt{\mu_1^2 - \beta^2\nu^2}$$

$$H_{11}^{1} = -\frac{\beta^2}{6\lambda}F_{04}^{0} + \frac{2\beta^2}{3}F_{13}^{0} - \frac{\beta^2\lambda}{2}F_{22}^{0} - \frac{16\beta^2\sigma^{1/2}}{3\lambda}\left[\lambda^2\nu^2\mu_1\sqrt{\mu_1^2 - \beta^2\nu^2} + \nu^4\cosh^{-1}\frac{\mu_1}{\beta\nu}\right]$$

$$H_{03}^{1} = \frac{\nu}{8\lambda}F_{04}^{1} - \frac{16\beta^2\sigma^{1/2}\nu^2}{3}\left[\lambda^3\nu\mu_1^2 + \lambda\nu^3(2\sigma+1)\right]\sqrt{\mu_1^2 - \beta^2\nu^2}$$

As may be suspected from the complicated results generically represented by E_{mn}, F_{mn}^{l}, G_{mn}, and H_{mn}^{l}, the section lift and moment coefficients are not simplified to any

large extent by algebraic combination. The lift and moment coefficients, as given in Eqs. 5-14 are in a form which is perhaps better for numerical evaluation than a combined form.

Some cautionary remarks are in order with regard to Eqs. 5-13 and 5-14. In evaluating the coefficients L_i and M_i (where $i = 3,4$), the terms involving k^2 should be ignored. This results from the fact that the lift and moment coefficients associated with α have been treated completely only to the same order as those associated with $\dot{h}/U$. In Eqs. 5-14 only real quantities are admitted. Therefore, the following restrictions must be noted:

$$\left.\begin{aligned} \sqrt{\mu_1^2 - \beta^2\nu^2} &= 0 \\ \cos^{-1}\frac{\mu_1 + \beta^2\lambda\nu}{\beta(\lambda\mu_1 + \nu)} &= 0 \\ \cos^{-1}\frac{\mu_1 - \beta^2\lambda\nu}{\beta(\lambda\mu_1 - \nu)} &= \pi \\ \cosh^{-1}\frac{\mu_1}{\beta\nu} &= 0 \end{aligned}\right\} \text{for } \frac{\nu}{\mu_1} \geq \frac{1}{\beta}$$

Total force and moment coefficients. As indicated in Eqs. 5-8 and 5-9, the total lift and moment coefficients, $\bar{L}_i$ and $\bar{M}_i$, may be obtained from the previously derived L_i and M_i by setting $\mu_1 = 1$ in L_i and M_i and integrating the resulting expressions over the span of the wing. The results of the spanwise integration combine readily to yield expressions which are much simpler than the corresponding ones in the section coefficient case. Without further detail, the total force and moment coefficients are:

$$\bar{L}_1 + i\bar{L}_2 = \bar{L}_1' + i\bar{L}_2'$$

$$\bar{L}_3 + i\bar{L}_4 = \bar{L}_3' + i\bar{L}_4' - \left(\frac{i}{k} + 2\mu_0\right)(\bar{L}_1' + i\bar{L}_2') \qquad (5\text{-}15)$$

$$\bar{M}_1 + i\bar{M}_2 = \bar{M}_1' + i\bar{M}_2' - 2\mu_0(\bar{L}_1' + i\bar{L}_2')$$

$$\bar{M}_3 + i\bar{M}_4 = \bar{M}_3' + i\bar{M}_4' - 2\mu_0(\bar{L}_3' + i\bar{L}_4') - \left(\frac{i}{k} + 2\mu_0\right)(\bar{M}_1 + i\bar{M}_2)$$

where

$$\bar{L}_1' = \frac{2\lambda}{3\beta^3} - \frac{M^2\lambda}{15\beta^7}(4M^2+1)k^2$$

$$\bar{L}_2' = \frac{\lambda}{\beta k} - \frac{M^2\lambda}{2\beta^5}k$$

$$\bar{L}_3' = \frac{2\lambda}{3\beta^3}$$

$$\bar{L}_4' = \frac{4\lambda}{3\beta k} - \frac{2M^2\lambda}{5\beta^5}k$$

$$\bar{M}_1' = \frac{\lambda}{\beta^3} - \frac{M^2\lambda}{9\beta^7}(4M^2+1)k^2$$

$$\bar{M}_2' = \frac{4\lambda}{3\beta k} - \frac{4M^2\lambda}{5\beta^5}k$$

$$\bar{M}_3' = \frac{16\lambda}{15\beta^3}$$

$$\bar{M}_4' = \frac{2\lambda}{\beta k} - \frac{2M^4\lambda}{3\beta^5}k$$

If Eqs. 5-10 and 5-11 had been put in the conventional form,

$$L = C_L q S$$

$$M = C_M q S c$$

where S is the area of the wing (for the triangle $s = 4\,b^2\lambda$), then the lift and moment coefficients,

similar to Eqs. 5-15, would be independent of λ. This is a rather remarkable result, since it means that the lift and moment coefficients for a triangular wing with supersonic edges are independent of the vertex angle of the triangle. The same observation was previously noted by Miles[2].

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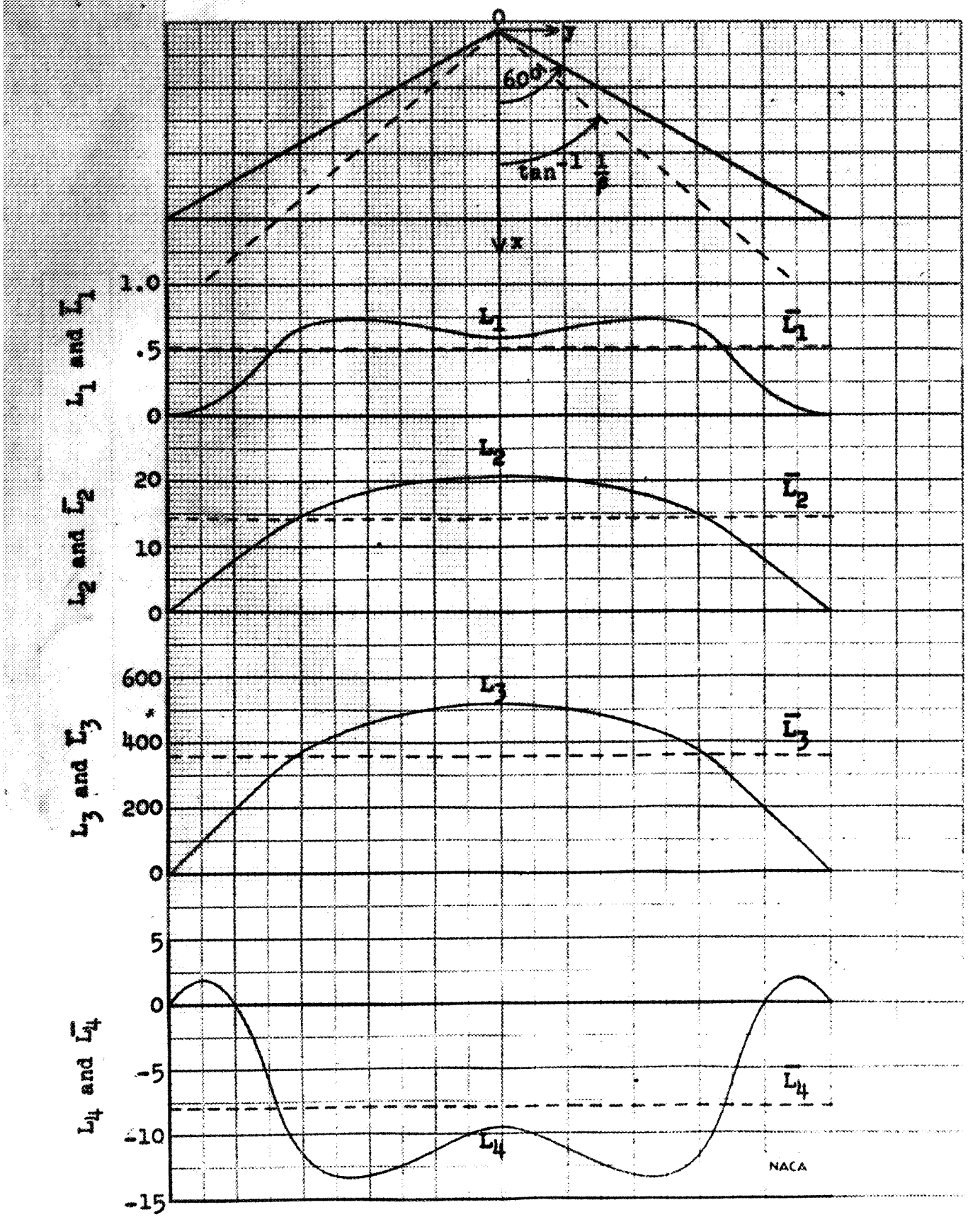
[2] Miles, op. cit.

CHAPTER VI

SOME CALCULATIONS AND DISCUSSION

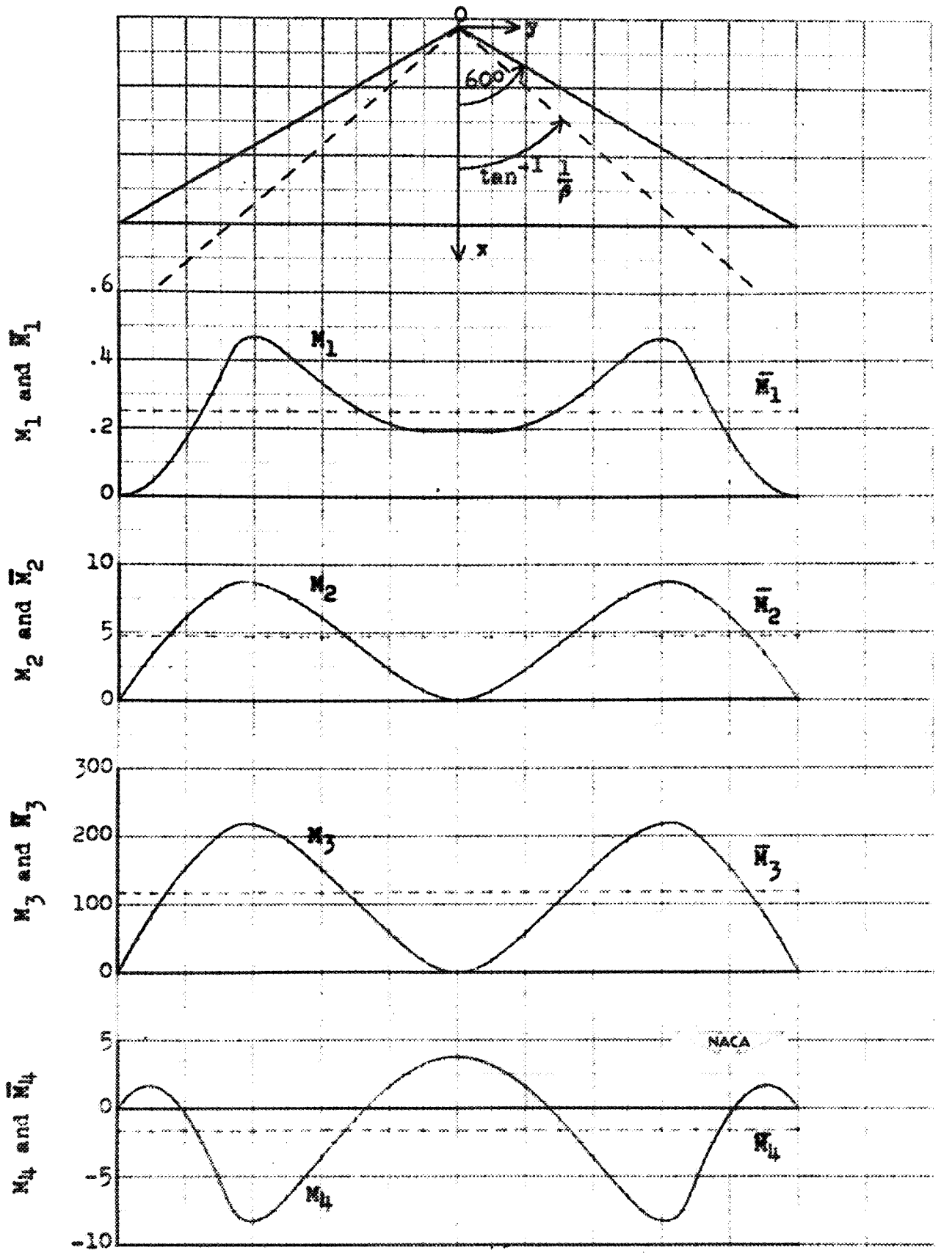
It may be of interest to the reader to examine the spanwise distributions of the various components of the section lift and moment coefficients for a particular case. Thus, the components of Eqs. 5-5 and 5-6, given more fully in Eqs. 5-12, have been evaluated at different spanwise positions, y, for plan form (a) of Fig. 3, for the following set of conditions: $\lambda = \sqrt{3}$, $\mu_o = 0.5$, $M^2 = 1.75$, and $k = 0.04$. These sample results are plotted as functions of the span position in Fig. 4. The spanwise variations of the section force components are shown in Fig. 4(a) and the corresponding variations of the section moment coefficients in Fig. 4(b).

The reader may also find interesting a comparison of the total force and moment coefficient components and the previous results. The total components, given by Eqs. 5-15 when referred to the nondimensional span of the wing 2λ, should represent the average ordinates of the corresponding section component distributions. The components of the total force and moment coefficients have been evaluated for the same set of conditions previously used. The results are represented by dashed lines in Fig. 4. It may be noted in Fig. 4 that the areas under the dashed lines are equal to those under



(a) Lift force

Figure 4.- Spanwise distribution of components of lift force and moment coefficients for $\mu_0 = 0.5$, $\lambda = \sqrt{3}$, $\kappa^2 = 1.75$, and $k = 0.04$.



(b) Moment

Figure 4.- Concluded.

the section component curves, as should be the case. As a result of the linear dependence of the total force and moment coefficients on the semi apex angle of the wing, as may be noted in Eqs. 5-15, the average ordinates of the distributions given in Fig. 4 will not change if only the apex angle is changed. Only the shapes of the distributions will be affected by such a variation in angle.

In the example plotted in Fig. 4 the total component of moment coefficient $\bar{M}_4$, is negative. This term would therefore not contribute to the aerodynamic damping but would act as a source of energy for the oscillating system. This result is significant since it indicates the possibility of single-degree torsional flutter.

Discussion of $\bar{M}_4$. The wing plan forms discussed in this paper exhibit the possibility of loss of aerodynamic damping in torsion for certain ranges of the Mach number, M, and axis of rotation location, μ_o. This is indicated, as mentioned in the previous paragraph, by the negative value of the torsional damping moment component, $\bar{M}_4$, in the example given. Since a wing will oscillate as a rigid body at a fairly low frequency of oscillation, the main results of the loss of damping phenomenon can be obtained by maintaining $\bar{M}_4$ to the order of 1/k only. When this is done there is obtained for plan form (a) of Fig. 3, the following result from Eq. 5-15

$$\bar{M}_4 = \frac{\lambda}{3\beta^3 k}\left[12(M^2-1)\mu_o^2 - 4\mu_o(4M^2-5)+3(2M^2-3)\right] \qquad (6\text{-}1)$$

In general, torsional stability depends on the sign and magnitude of $\bar{M}_4$. Positive values of $\bar{M}_4$ indicate stability and negative values indicate possible instability. The borderline case is thus given by $\bar{M}_4 = 0$.

The ranges of values of Mach number, M, and axis of rotation, μ_o, for which $\bar{M}_4$ in Eq. 6-1 vanishes are shown in Fig. 5. Only one curve is obtained for all delta wings with supersonic edges, that is $\beta\lambda \geq 1$, since $\bar{M}_4$ is dependent on λ for magnitude alone, as may be noted in Eq. 6-1. The region inside the curve in Fig. 5 is the region of possible instability. For convenience a second ordinate is given in Fig. 5 showing the values of λ below which the leading edges become subsonic. With the aid of this second ordinate it may be seen that a delta wing with a semi vertex angle of 45° is the narrowest triangular wing that may show a torsional instability and yet retain the characteristic $\beta\lambda \geq 1$, since the uppermost part of the curve in the figure corresponds to a value of $\lambda = 1$ when $\beta\lambda = 1$. For a narrower wing to have at least sonic edges it must be moving at a Mach number which would place it in the stable region of Fig. 5.

CHAPTER VII

SUMMARY

This thesis contains a derivation of the lift and moment on a thin, oscillating, arrow-head type wing with supersonic edges. In the derivation a linearized form of Bernoulli's equation is used to find the lift and moment from the velocity potential for the wing. The development of this velocity potential constitutes a major portion of this thesis. Although the velocity potential that is found is not necessarily restricted to harmonic wing motions, explicit results for the lift and moment are given only for the wing undergogin harmonic torsional and vertical oscillations.

Closed expressions for the velocity potential, section force and moment coefficients for the "general" arrow-head wing, and total force and moment coefficients for the particular arrow-head wing known as the "wide" delta are developed to the third power of the frequency of oscillation. These expressions are considered sufficient for a large number of practical applications. The procedure for obtaining terms involving higher powers of the frequency than the third is straightforward, but the calculations involved become increasingly laborious.

From an examination of the expressions for the forces and moments it is found that the arrow-head wing exhibits negative aerodynamic damping in torsion for certain ranges

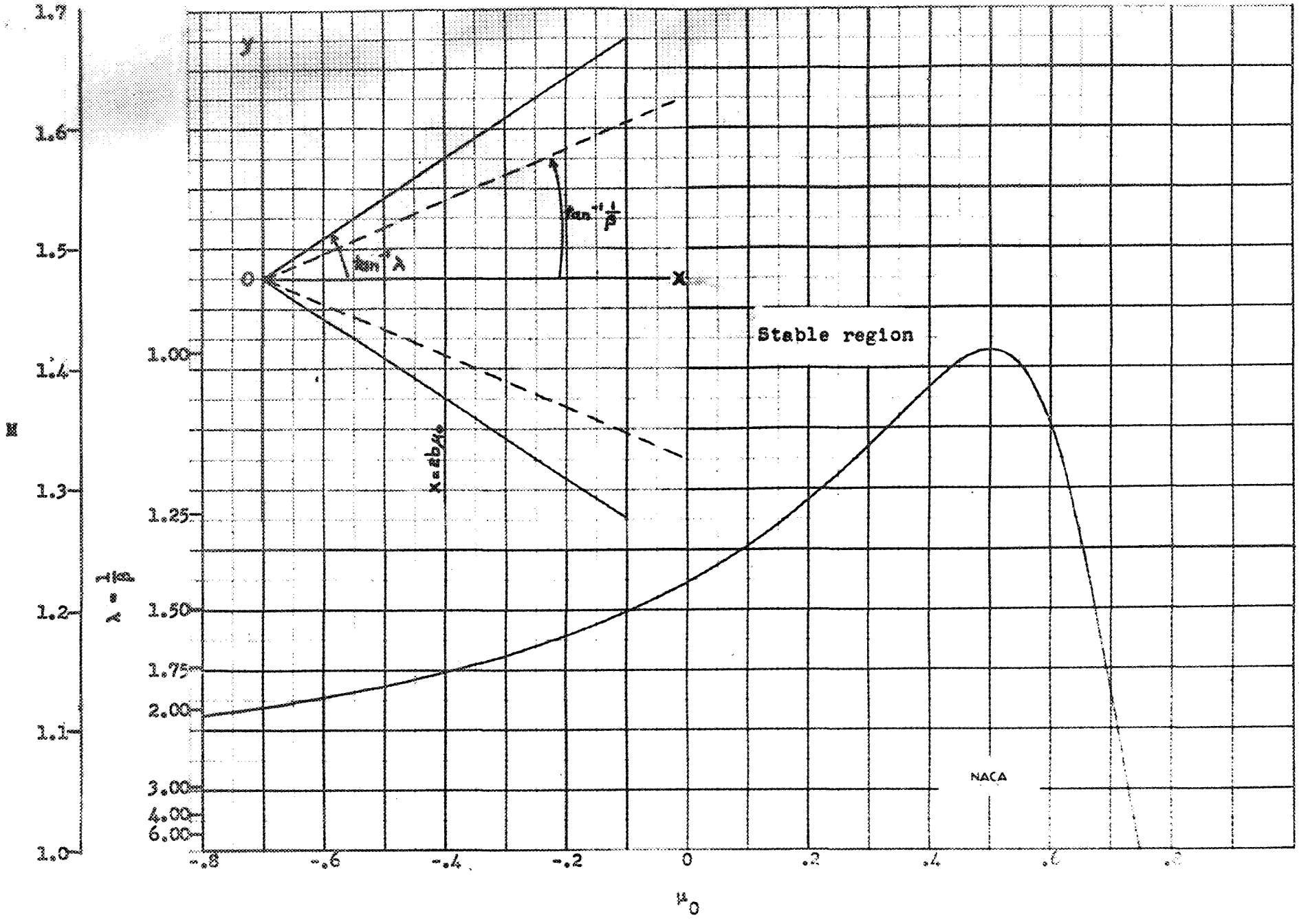


Figure 5.- Curve showing ranges of M and $\lambda = \frac{1}{\beta}$ against μ_0 for which the aerodynamic torsional

of Mach number and axis of rotation position. Thus, the possibility of single-degree torsional flutter is indicated for this type of wing. A figure showing the ranges of these parameters for the delta wing is given herein. This figure contains only one curve indicating that the loss of damping phenomenon is not dependent on the semi-apex angle of the delta wing with supersonic edges.

Langley

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